最小生成树
Minimum Spanning Trees

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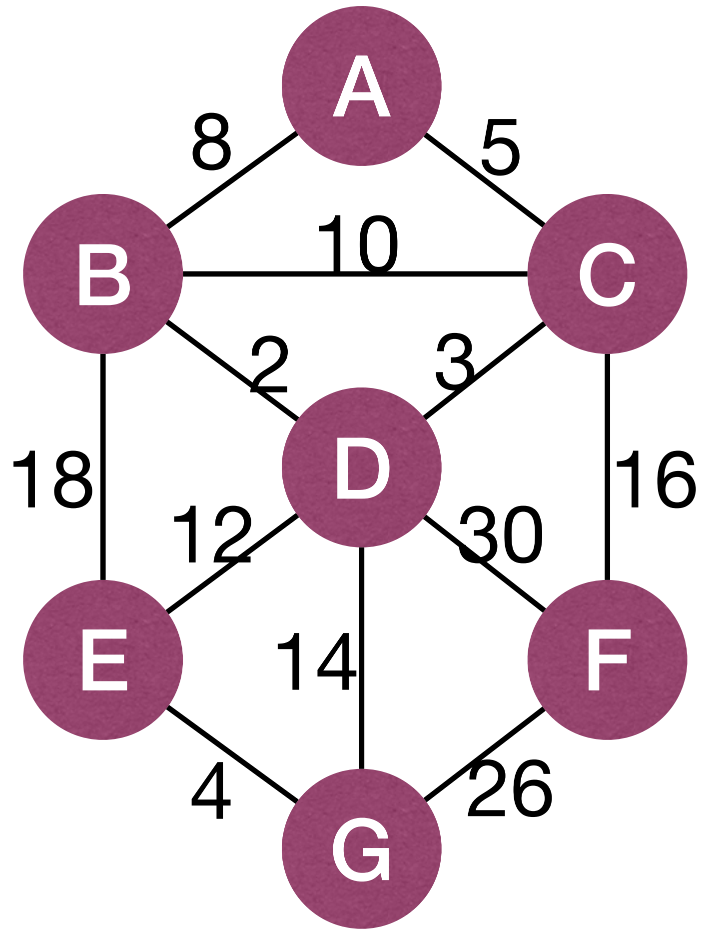
Minimum Spanning Trees (MST)

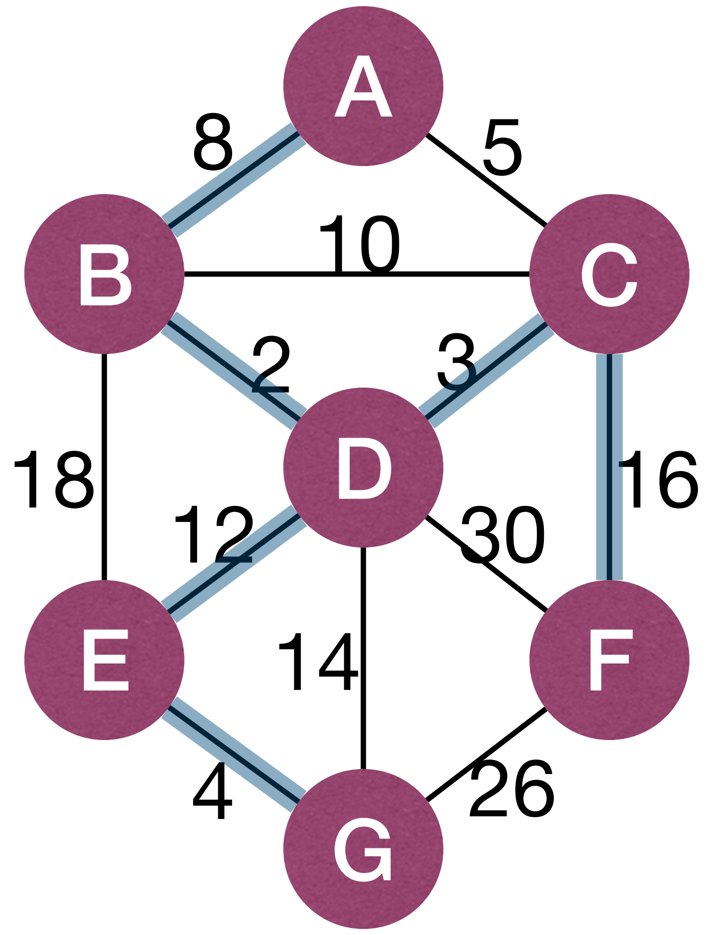
- Consider a connected, undirected, ***weighted*** graph G.

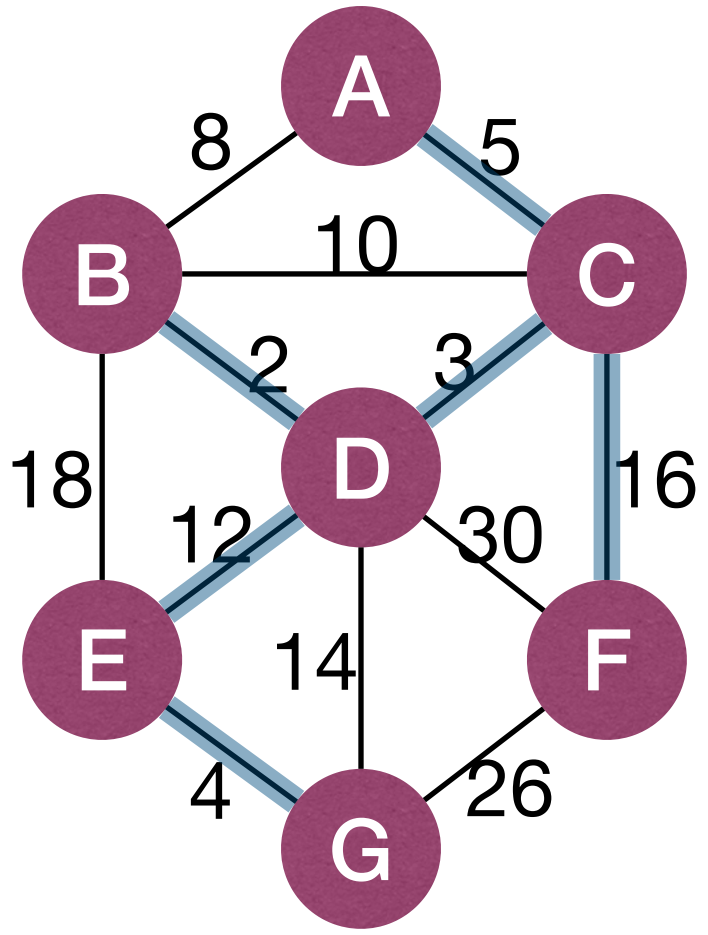
- That is, we have a graph $G = (V, E)$ together with a weight function $w : E \to \mathbb{R}$ that assigns a real weight $w(u, v)$ to each edge $(u, v) \in E$.

- A spanning tree is a tree containing **all** nodes in V and a subset T of all the edges E.

- A minimum spanning tree (MST) is a spanning tree whose total weight $w(T) = \sum_{(u,v)\in T} w(u, v)$ is minimized.







Application of MST

- Network Design:
  - E.g., build a minimum cost network connecting all nodes.
    - Transportation networks.
    - Water supply networks.
    - Telecommunication networks.
    - Computer networks.
- Many other applications…
  - E.g., important subroutine in more advanced algorithms.
    - One such application is used in a classical approximation algorithm for solving TSP.

Computing MST

- Consider the following generic method:
  - Starting with all nodes and an empty set of edges A.
  - Find some edge to add to A, maintaining the loop invariant that "A is a subset of some MST". (At anytime, A is the edge set of a **spanning forest**.)
  - Repeat above step until we have a spanning tree. (The resulting spanning tree must be a **MST**.)

These edges are called "**safe edges**", how to identify them?

Easy to determine, e.g., $|A| = n - 1$

GenericMST(G,w):

$A := \varnothing$

while *A is **not** a spanning tree*

  $(u,v) :=$ *find_a_edge_maintaining_the_loop_invariant*()

  $A := A \cup \{(u, v)\}$

return A

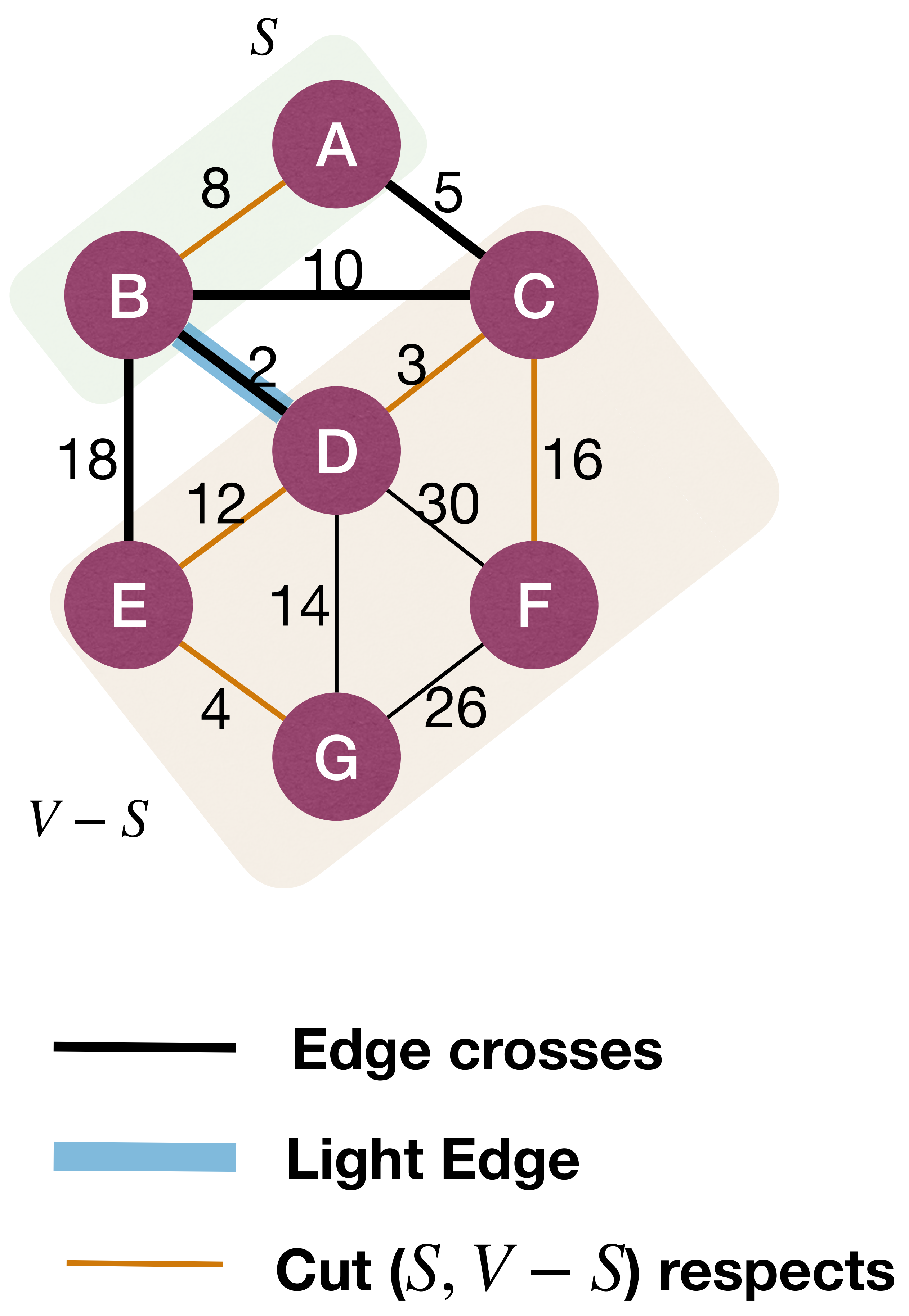
Identifying Safe Edges

- A **cut** $(S, V - S)$ of $G = (V, E)$ is a partition of V into two parts.

- An edge **crosses** the cut $(S, V - S)$ if one of its endpoint is in S and the other endpoint is in $V - S$.

- A cut **respects** an edge set A if no edge in A crosses the cut.

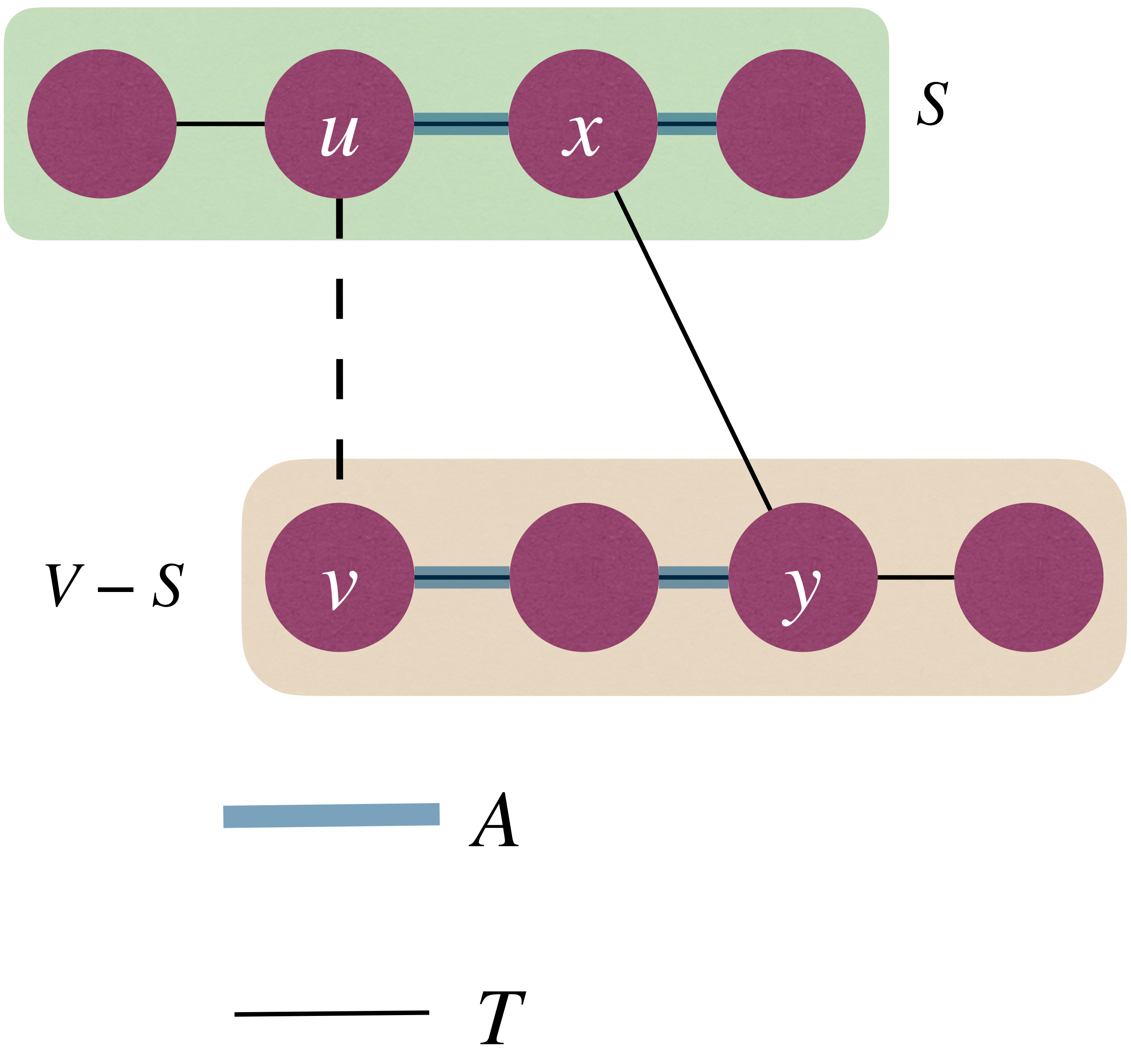
- An edge is a **light edge** crossing a cut if the edge has minimum weight among all edges crossing the cut.



Identifying Safe Edges

Theorem [Cut Property] Assume A is included in the edge set of some MST, let $(S, V-S)$ be any cut respecting A. If (u, v) is a light edge crossing the cut, then (u, v) is safe for A.

- Proof:
  - Let T be an MST containing A, assume T does not include (u, v).
  - Connecting (u, v) forms a cycle in T, and in that cycle some edge other than (u, v) crosses the cut. Let $(x, y) \in T$ be that edge.
  - $T' = T - (x, y) + (u, v)$ must be a spanning tree.
  - Since (u, v) is a light edge crossing the cut, T' must be an MST, and (u, v) is safe for A in T'.



Computing MST

Theorem [Cut Property] Assume A is included in the edge set of some MST, let $(S, V-S)$ be any cut respecting A. If (u, v) is a light edge crossing the cut, then (u, v) is safe for A.

GenericMST(G,w):

$A := \varnothing$
while A *is* ***not*** *a spanning tree*
    $(u,v) :=$ ***find_a_safe_edge***()
    $A := A \cup \{(u, v)\}$
return A

Corollary Assume A is included in some MST, let $G_A = (V, A)$. Then for any connected component in G_A, its **minimum-weight-outgoing-edge** (MWOE) in G is safe for A.

In G_A, an edge in a CC is "outgoing" if it connects to another CC

Kruskal's Algorithm

- Cut property: Assume A is included in some MST, let $G_A = (V, A)$. Then for any connected component in G_A, its MWOE in G is safe for A.

- Strategy for finding safe edge in Kruskal's algorithm: Find minimum weight edge connecting two CC in G_A.

Joseph Kruskal

<u>KruskalMST(G,w):</u>

$A := \emptyset$
Sort edges into weight increasing order
for each *edge* (u,v) *taken in weight increasing order*
  if *adding edge* (u,v) *does not form cycle in* A
    $A := A \cup \{(u, v)\}$
return A

- **Put another way**:
  - Start with n CC (each node itself is a CC) and $A = \emptyset$.
  - Find minimum weight edge connecting two CC. (# of CC reduced by 1.)
  - Repeat until one CC remains.

Kruskal's Algorithm

- Eden weights in increasing order: 2 3 4 5 8 10 12 14 16 18 26 30

Kruskal's Algorithm

KruskalMST(G,w):
$A := \varnothing$
Sort edges into weight increasing order
for each *edge* (u,v) *taken in weight increasing order*
    if ***adding edge (u,v) does not form cycle in A***
        $A := A \cup \{(u, v)\}$
return A

- How to determine an edge forms a cycle?
  - Put another way, how to determine if the edge is connecting two CC?

Use disjoint-set data structure!

Each set is a CC, u and v in same CC if: `Find(u) = Find(v)`.

Kruskal's Algorithm

<u>KruskalMST(G,w):</u>

$A := \varnothing$

Sort edges into weight increasing order → $O(m \log m) = O(m \log n)$ ($m \leq n^2$)

for *each node u in V*
    MakeSet(u)
→ $O(n)$

for each *edge* (u,v) *taken in weight increasing order*
    if $Find(u) \mathrel{!=} Find(v)$
        $A := A \cup \{(u, v)\}$
        Union(u, v)
→ $O(m \log^* n)$

return A

- Runtime of Kruskal's algorithm?
  - $O(m \log n)$ when using disjoint-set data structure

Prim's Algorithm

- Strategy for finding safe edge in Prim's algorithm: Keep finding MWOE in one fixed CC in G_A.

PrimMST(G,w):

$A := \emptyset$

$C_x := \{x\}$

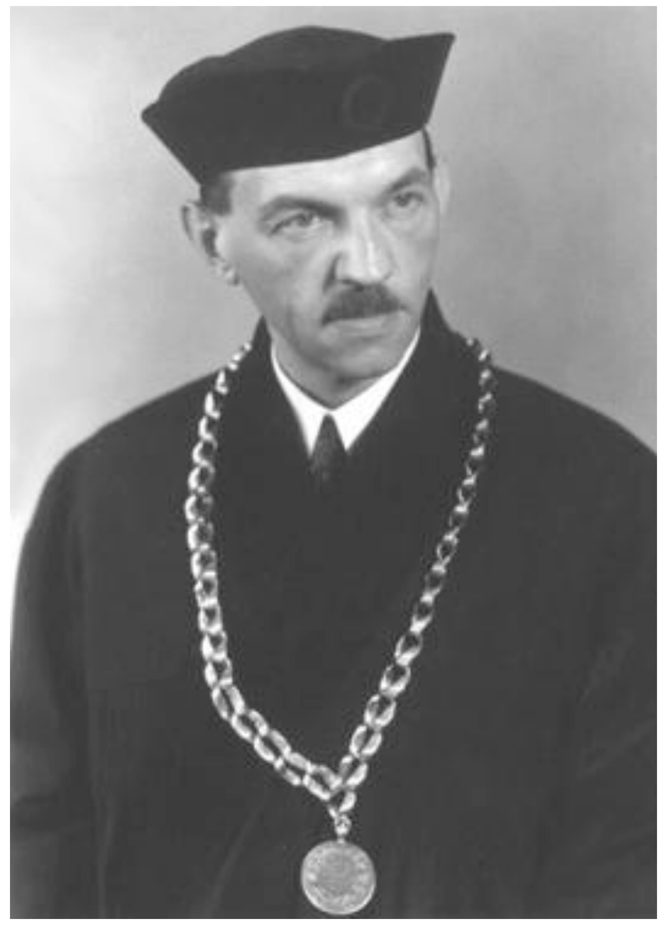
while C_x *is **not** a spanning tree*

  Find *MWOE* (u, v) of C_x

  $A := A \cup \{(u, v)\}$

  $C_x := C_x \cup \{v\}$

return A



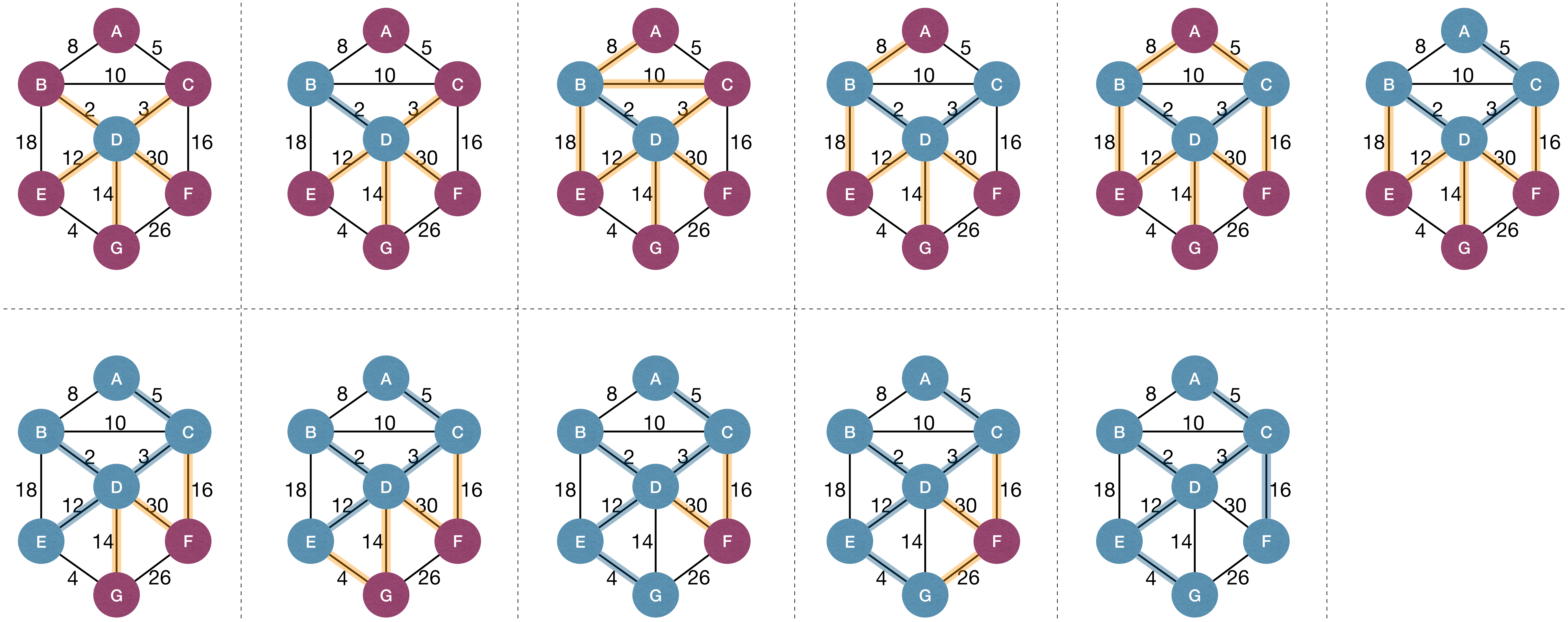
Vojtěch Jarník

Robert C. Prim

Edsger W. Dijkstra

- **Put another way**:
  - Start with n CC (each node itself is a CC) and $A = \emptyset$. Pick a node x.
  - Find MWOE of the component containing x (# of CC reduced by 1.)
  - Repeat until one CC remains.

Prim's Algorithm



Prim’s Algorithm

PrimMST(G,w):

$A := \emptyset$

$C_x := \{x\}$

while C_x *is* ***not*** *a spanning tree*

  Find MWOE (u, v) of C_x

  $A := A \cup \{(u, v)\}$

  $C_x := C_x \cup \{v\}$

return A

- How to find *MWOE* efficiently?
- **Put another way**: how to find the next node that is closest to C_x?
  - Use a priority queue to maintain each remaining node’s distance to C_x.

Prim's Algorithm

PrimMST(G,w): → $O(m \lg n)$ using binary heap to implement priority queue

$x :=$ *Pick an arbitrary node in G*

for each *node u* **in** *V* → $O(n)$

  $u.dist := INF,\ u.parent := NIL,\ u.in := False$

$x.dist := 0$

PriorityQueue *Q := Build a priority queue based on "dist" values* → $O(n)$

while *Q is* **not** *empty*

  $u := Q.ExtractMin()$ → $O(n \lg n)$

  $u.in := True$

  for each *edge (u,v)* **in** *E* → $O(m \lg n)$

    if $v.in = False$ **and** $w(u,v) < v.dist$

      $v.parent := u,\ v.dist := w(u,v)$

      $Q.Update(v, w(u,v))$

Could be faster using better priority queue implementation (By using fibonacci heaps instead)

DFS, BFS, Prim, and others…

```
DFSIterSkeleton(G, s):
Stack Q
Q.push(s)
while !Q.empty()
    u := Q.pop()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.push(v)
```

```
BFSSkeletonAlt(G, s):
FIFOQueue Q
Q.enque(s)
while !Q.empty()
    u := Q.dequeue()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.enque(v)
```

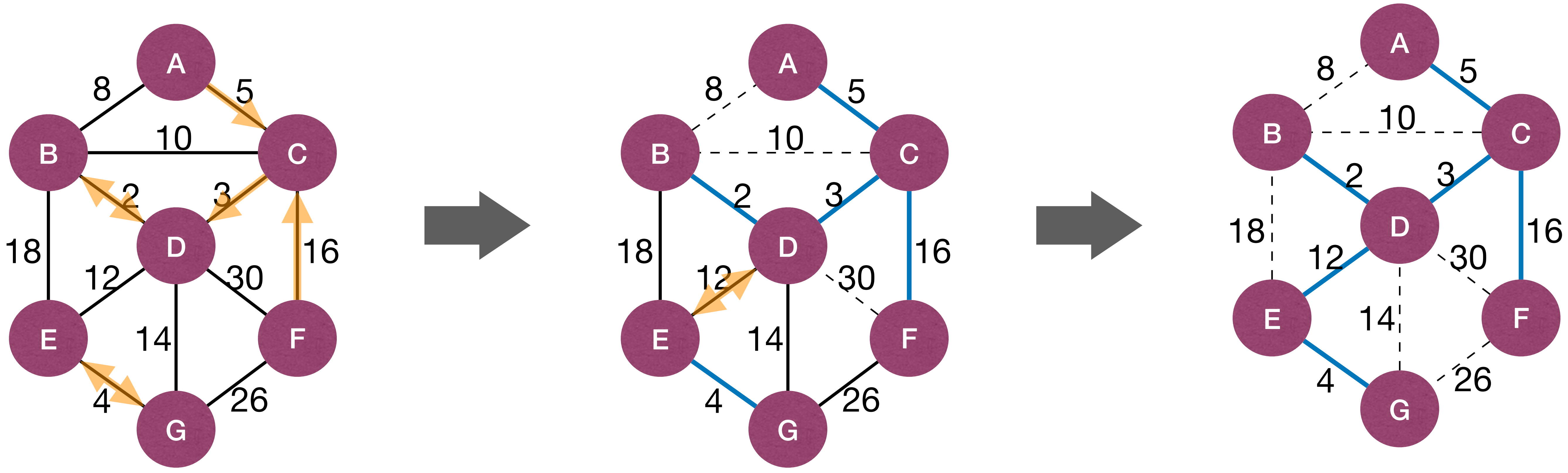
```
PrimMSTSkeleton(G, x):
PriorityQueue Q
Q.add(x)
while !Q.empty()
    u := Q.remove()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            if !v.visited and …
                Q.update(v, …)
```

```
GraphExploreSkeleton(G, s):
GenericQueue Q
Q.add(s)
while !Q.empty()
    u := Q.remove()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.add(v)
```

Borůvka's Algorithm

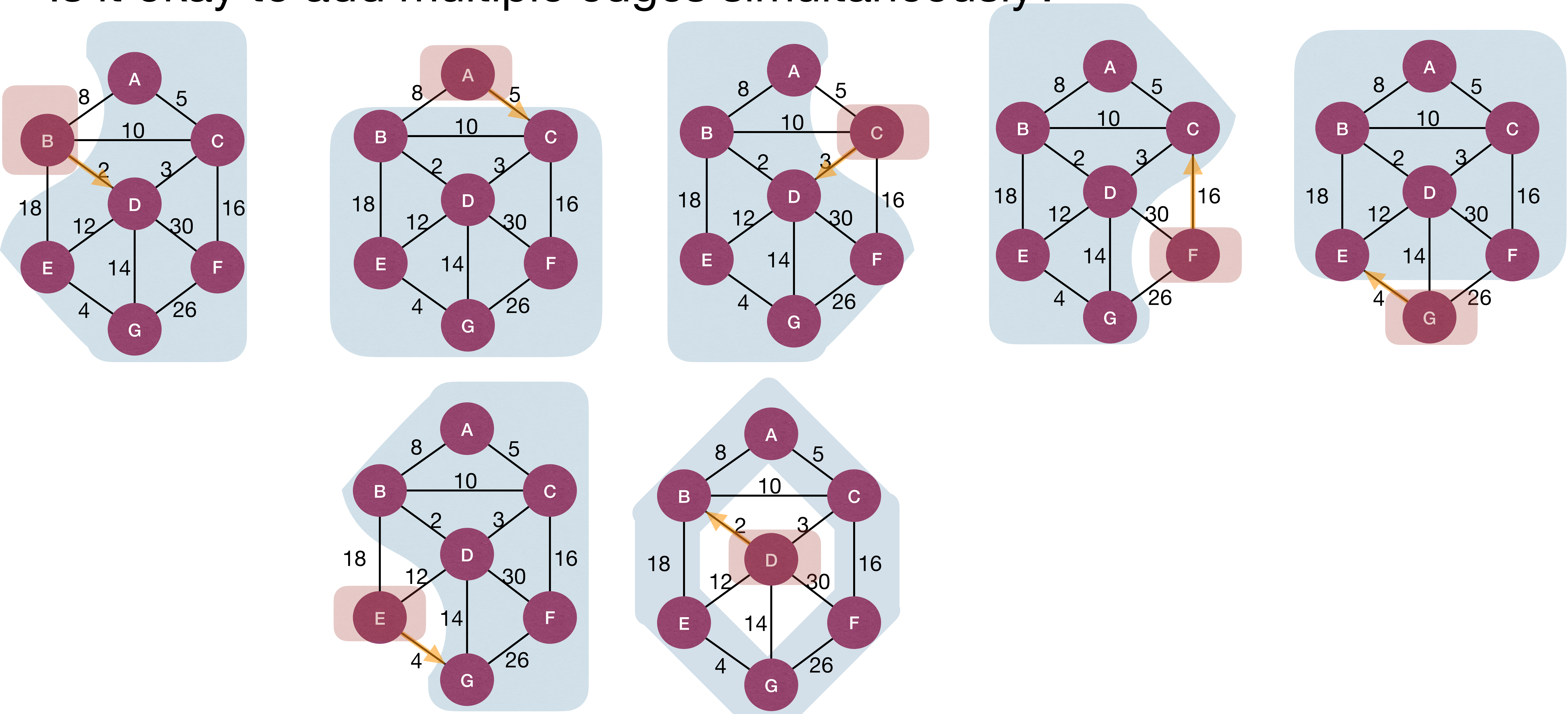
- Borůvka's algorithm for computing MST (actually the <u>earliest</u> MST algorithm):
  - Starting with all nodes and an empty set of edges A.
  - Find MWOE for every remaining CC in G_A, add all of them to A.
  - Repeat above step until we have a spanning tree.

Otakar Borůvka



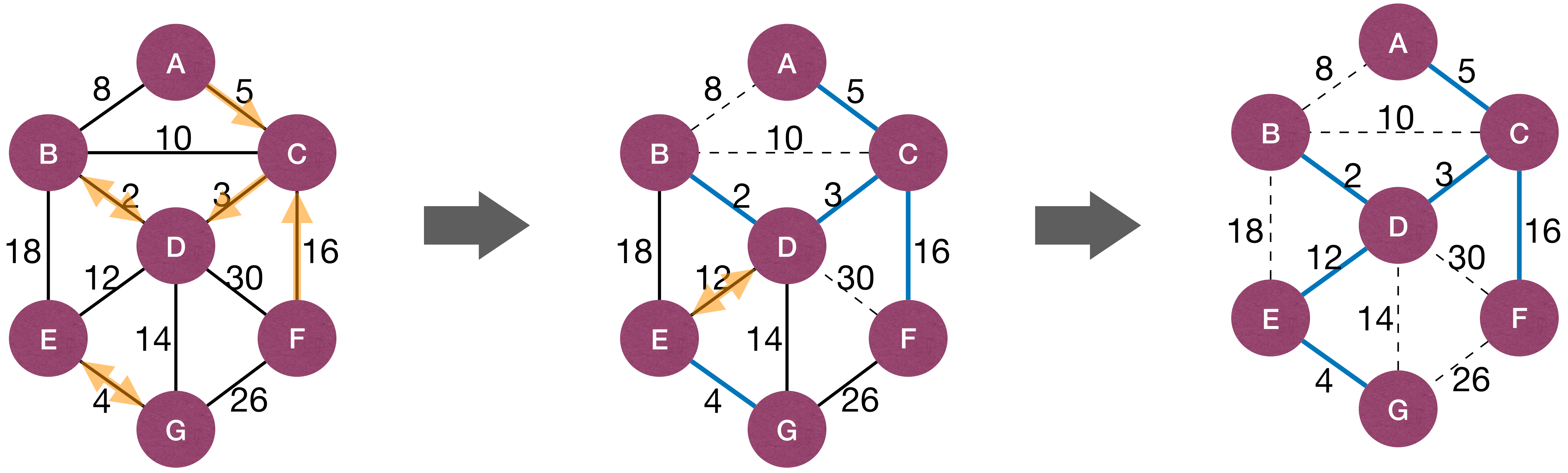
Borůvka's Algorithm

- Is it okay to add multiple edges simultaneously?



Borůvka's Algorithm

- Is it okay to add multiple edges simultaneously?
- But it may result in circles?
    - Assuming all edge weights are distinct, if CC C_1 propose MWOE e_1 to connect to C_2, and C_2 proposes MWOE e_2 to connect to C_1, then $e_1 = e_2$.



Borůvka's Algorithm

KruskalMST(G,w): → Total runtime is $O(m \lg n)$

$G' := (V, \varnothing)$

do

  $ccCount := CountCCAndLabel(G')$ → $O(n)$ //Do DFS/BFS, count #of CC, give **ccNum** to nodes. (belong to the $ccNum^{th}$ CC)

  for $i := 1$ **to** $ccCount$

    $safeEdge[i] := NIL$

  → $O(n)$

  for **each** *edge* (u,v) **in** $E(G)$

    if $u.ccNum \mathrel{!}= v.ccNum$

      if $safeEdge[u.ccNum] = NIL$ **or** $w(u,v) < w(safeEdge[u.ccNum])$

        $safeEdge[u.ccNum] := (u,v)$

      if $safeEdge[v.ccNum] = NIL$ **or** $w(u,v) < w(safeEdge[v.ccNum])$

        $safeEdge[v.ccNum] := (u,v)$

  → $O(m+n) = O(m)$

  for $i := 1$ **to** $ccCount$

    Add $safeEdge[i]$ **to** $E(G')$

  → $O(n)$

while $ccCount > 1$ → $O(\lg n)$ interactions ← WHY?

return $E(G')$

Borůvka’s Algorithm

- Why Borůvka’s algorithm is interesting?
  - The number of components in G' can drop by significantly more than a factor of 2 in a single iteration, reducing the number of iterations below the worst-case $O(\lg n)$.
  - Borůvka’s algorithm allows for parallelism naturally; while the other two are intrinsically sequential.
  - Generalizations of Borůvka’s algorithm lead to faster algorithms.

Summary

- The "Cut Property" leads to many MST algorithms: Assume A is included in some MST, let $(S, V - S)$ be any cut respecting A. If (u, v) is a light edge crossing the cut, then (u, v) is safe for A.

- Classical algorithms for MST, all with runtime $O(m \cdot \log n)$:
  - Kruskal (UnionFind): keep connecting two CC with min-weight edge.
  - Prim (PriorityQueue): grow single CC by adding MWOE.
  - Borůvka: add MWOE for all CC in parallel in each iteration.

- Can we do MST in $O(m)$ time?
  - Randomized algorithm with expected $O(m)$ runtime exists.

Further reading

- [CLRS] Ch.23
- [Erickson] Ch.7